Chapter 12

Projects

This chapter contains a collection of projects on single variable calculus. Students may work individually or in groups of up to four students. Some projects may take one week. Others may require two or three weeks.

Projects for Calculus I

- A Power Relay Station
- The Search for the Meteorite
- Speed Limits
- Terminal Velocity
- The Ant and the Blade of Grass
- Distance Between Two Curves
- Tangent and Normal Lines
- Parameterizing Letters
- Parametric Curves
- Seeing a Blimp
- Static and Dynamic Tension

Projects for Calculus II

- Calculus I Review
- Calibrating a Dipstick
- The Area of a Unit p-Ball
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- Center of Gravity of a Parabolic Plate

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- Area Between a Curve and Its Tangent Line

- Curves Generated by Rolling Circles

- The Wankel Rotary Engine

- Shakespeare's Shylock

- The Bouncing Ball

- Pension Funds

- The Flight of a Baseball

- Parachuting

- Radioactive Waste at a Nuclear Power Plant

- Visualizing Euler's Method

- The Brightest Phase of Venus

12.1   A Power Relay Station

BACKGROUND: Review setting up word problems and graphing functions.

ASSIGNMENT: On a long straight highway, there are 11 communities located at mile markers $1.8, 2.2, 3, 3.35, 6.1, 7.35, 7.5, 8.3, 10.6, 11.2, 13.5$. The electric company which serves these 11 communities would like to construct a power relay station somewhere along this highway. The company would then run separate power lines from the relay station to each of the 11 communities. To minimize the cost of constructing these 11 power lines, the company would like to locate the relay station so as to minimize the sum of the distances to each station.

1. Think of the communities as located along the x-axis at the mile markers that are given. Suppose the relay station is located at position x. Write an expression for the sum of the distances from the power station to the communities using Maple's `abs` (absolute value) command. Plot it.

2. Where along the highway should the electric company build its relay station? Justify your answer. Is this location unique?

3. Suppose a 12$^{\text{th}}$ community, located at mile 15.7, is added to the highway. Now where should the electric company locate its relay station? Is this location unique?

12.2 The Search for the Meteorite

BACKGROUND: Review the equation of an hyperbola. You will use `solve` to find the intersection of two hyperbolas.

ASSIGNMENT: A meteorite crashes somewhere in the hills that lie north of point A. The impact is heard at point A and, 8 seconds later, it is heard at point B, 6 miles due west of A. Six seconds still later it is heard at point C, 4 miles due east of A. Locate the point of impact of the meteorite. The speed of sound is roughly .20 miles per second. Include an accurate plot with your explanation.

HINT 1: The equation of the hyperbola centered at the origin with focal points at $(+c, 0)$ and $(-c, 0)$ along the x-axis, is given by

$$\frac{x^2}{a^2} - \frac{y^2}{c^2 - a^2} = 1$$

Here, $2a$ is the difference $|d_1 - d_2|$, where d_1 is the distance between an arbitrary point (x, y) on the hyperbola and the focal point $(+c, 0)$, and d_2 is the distance between (x, y) and the focal point $(-c, 0)$.
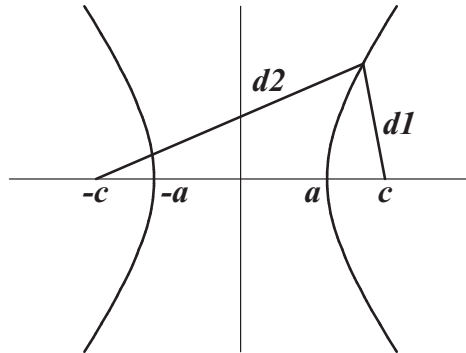
HINT 2: For this project, locate points A, B, and C on the x-axis. Place the origin at the midpoint between points A and B. From the information given, the meteorite must lie on an hyperbola centered at the origin with points A and B as focal points. Find the equation of this hyperbola. Likewise, the meteorite must also lie on an hyperbola with focal points at A and C. (Where is this hyperbola centered? What is its equation?) Thus, the meteorite must lie at the point of intersection of the two hyperbolas.

NOTE: An hyperbola has two branches. You must take care to choose the correct branch of each hyperbola.

12.3   Speed Limits

GOAL: The goal of this project is to learn about average and instantaneous velocities. In the process you will use Maple to define, evaluate and plot functions and compute limits.

BACKGROUND: Before you start, try each of the following examples.

- Definition and evaluation of functions.

  The following Maple command defines the function $x(t) = \dfrac{t^3 - 9t}{3t^3 - 9t^2 + t - 3}$, which might be interpreted as the position of an object at time t.

  ```
  >   x:=t -> (t^3-9*t)/(3*t^3-9*t^2+t-3);
  ```

  Then the position at $t = 1$ is

  ```
  >   x(1);
  ```

  You can also define functions of two or more variables. For example, the following gives the average velocity of the object between the times $t = t_1$ and $t = t_2$.

  ```
  >   AveVel:=(t1,t2) -> (x(t2)-x(t1))/(t2-t1);
  ```

  Then the average velocity between $t = 4$ and $t = 5$ is

  ```
  >   AveVel(4,5);
  ```

- Plots.

  The function $x(t)$ defined above, can be plotted on the interval $0 \le t \le 10$ by using the Maple command

  ```
  >   plot(x(t),t=0..10);
  ```

- Limits.

  Notice that the function $x(t)$ is undefined at $t = 3$ because the denominator is zero there. Maple agrees.

  ```
  >   x(3);
  ```

  ```
  Error, (in x) numeric exception: division by zero
  ```

  However, Maple can compute the limit as t goes to 3.

  ```
  >   Limit(x(t),t=3);    value(%);    evalf(%);
  ```

  You should be able to verify this limit, both in the plot and by algebraically manipulating the formula for $x(t)$.

ASSIGNMENT: A Texas Aggie suddenly decided to take his girlfriend to Houston for the day. So he hopped in his car and started to drive from College Station to Houston. When he got to Navasota, he suddenly realized that he had forgotten his girlfriend. Embarrassed, he slowly drove back to College Station and got his girlfriend. All this took 2 hours. Then they proceeded to Houston, taking another 2 hours. His distance (in miles) from College Station at time t (in hours) was $x(t) = \dfrac{200t^2(t-2)^2}{3 + 9t^2}$.

1. Enter his position function $x(t)$ into Maple using an arrow definition.

2. Plot his position $x(t)$ for times between $t = 0$ when he first left College Station and $t = 4$ when he finally arrived in Houston.

3. Approximately when did he arrive in Navasota? Approximately, what is the distance from College Station to Navasota?
   HINT: Click in the plot window at the first maximum and read off the t and x coordinates (at the left end of the toolbar).

4. What is the exact distance from College Station to Houston?
   HINT: Compute $x(4)$.

5. Find his average velocity (Use `AveVel` defined above.)
   (a) from College Station to Navasota, the first time out. (approximate).
   (b) from Navasota back to College Station. (approximate).
   (c) from College Station to Houston, the second time out. (exact).
   (d) from College Station to Houston, the whole trip. (exact).

   NOTE: To avoid round-off error in the next problem, execute
   > `Digits := 20;`

6. Find his average velocity between time $t = 3$ and time $t = 3 + h$ for the following values of h

1	.1	.01	.0001
-1	$-.1$	$-.01$	$-.0001$

7. Use the values calculated above to guess his instantaneous velocity at $t = 3$.

8. (Clear `h` by executing `h:='h';`) Compute his average velocity between time $t = 3$ and time $t = 3 + h$ for a variable value of h. Use this formula to compute his instantaneous velocity at $t = 3$ by computing the limit

$$\lim_{h \to 0} AveVel(3, 3 + h)$$

9. Recompute the limit above by algebraically manipulating $AveVel(3, 3+h)$ using the `expand` and/or `factor` commands until the h in the denominator cancels. Then substitute in $h = 0$ using the `subs` command.

10. *10% extra credit.* At what time does he pass through Navasota on his second time out? HINT: If X is the distance to Navasota, solve the equation $x(t) = X$ using Maple's `fsolve` command.

12.4   Terminal Velocity

GOAL: The goal of this lab is to learn about asymptotic (or terminal) velocity.
BACKGROUND: You will use Maple to define, evaluate and plot expressions and
compute limits at infinity. Before you start, try each of the following examples.

- Definition and evaluation of expressions.

  The following Maple command defines the expression $x = \dfrac{t^3 - 9t}{3t^3 - 9t^2 + t - 3}$,
  which might be interpreted as the position of an object at time t.

  ```
  >  x:=(t^3-9*t)/(3*t^3-9*t^2+t-3);
  ```

  Then the position at $t = 1$ is

  ```
  >  eval(x, t=1);
  ```

- Plots.
  The expression x defined above, can be plotted on the interval $0 \le t \le 10$
  by using the Maple command

  ```
  >  plot(x,t=0..10);
  ```

- Limits.
  Notice that the expression x appears to approach a value of about $x \approx .5$
  as t gets large. Maple can compute this limit precisely:

  ```
  >  Limit(x,t=infinity);    value(%);
  ```

  Notice the limit is not .5! This is because the plot did not look far enough
  out in t. Correct the plot by changing the range for t. You should also be
  able to verify this limit by algebraically manipulating the formula for x.

ASSIGNMENT: A parachute jumper's velocity is $v = \dfrac{100t + 85t^2}{1 + 23t^2}$ in meters per
second where t is in seconds.

1. Enter her velocity v into Maple as an expression.

2. Plot her velocity v for the first 3 seconds of her fall and again for times
   between $t = 0$ (when she jumps) and $t = 120$ seconds (when she lands).

3. Find an approximate value for her maximum velocity.
   HINT: Click in the plot window on the maximum and read off the t and
   v coordinates (at the left end of the toolbar).

4. Find an approximate value for her terminal velocity.

5. Find an exact value for her asymptotic velocity by computing $\lim\limits_{t \to \infty} v$.
   NOTE: You should be able to verify this limit by algebraically manipulat-
   ing the formula for v.

6. Her landing will be safe if her velocity is less than 5 meters per second.
   Was her landing safe?

12.5   The Ant and the Blade of Grass

GOAL: In this project, you will use D to differentiate a function, construct a tangent line at a specific point and at a general point, plot a series of points connected by line segments, plot several things on the same graph, and solve an equation using fsolve.

BACKGROUND: Before you start, try each of the following examples.

- Differentiating functions.
  If a function f is defined using an arrow definition, for example

    ```
    >   f := x -> x^3;
    ```

  then its derivative is computed using Maple's D operator:

    ```
    >   D(f);
    ```

  Notice that the output is an arrow-defined function but it doesn't have a name. If you wish to give it a name, say Df, then you type

    ```
    >   Df:=D(f);
    ```

- Tangent lines.
  The line tangent to the curve $y = f(x)$ at the point $(a, f(a))$, is $y = f_{\tan}(x) = f(a) + f'(a)(x - a)$. For the function $f(x) = x^3$ discussed above, the tangent line at $x = 2$ can be defined as a function using:

    ```
    >   a:=2;
    >   ftan := x -> f(a) + Df(a)*(x-a);
    ```

  The equation of the tangent line is then

    ```
    >   y = ftan(x);
    ```

  HINT: The benefit of defining a:=2; instead of just typing 2 each time, is that if you need to compute the tangent line at several points, you can cut and paste the whole sequence of commands and only change the value of a before re-executing.

- Plotting several functions.
  The function $y = f(x)$ and its tangent line $y = f_{\tan}(x)$, defined above, can be plotted together on the interval $-5 \leq x \leq 5$ by enclosing the two functions in square brackets [ ] and using the Maple plot command.

    ```
    >   plot([f(x),ftan(x)], x=-5..5);
    ```

- Lists and plotting points.
  In Maple an ordered list is denoted by separating the items by commas and enclosing the list in square brackets [ ]. Thus, the point $(2, 5)$ is entered as [2,5], while a list of points might be

    ```
    >   octagon:=[ [1,0], [4,0], [5,1], [5,4], [4,5], [1,5],
    >   [0,4], [0,1], [1,0] ];
    ```

  If you plot this list of points and connect the dots, you should get an octagon.

```
>  plot(octagon);
```

If you have several lists of points, enclose them in square brackets [ ]. Try the following.

```
>  line1:=[ [1,3], [2,3] ]; line2:=[ [3,3], [4,3] ];
>  you:=[ [1,2], [2,1], [3,1], [4,2] ];
>  plot([octagon, line1, line2, you]);
```

What did you get?

Finally you can even plot lists of points together with functions and also specify the x-range, y-range and colors. For example

```
>  plot([f(x), ftan(x), octagon, you], x=0..5, y=0..6,
>  color=[red,blue,green,magenta]);
```

- Solving equations.
  Suppose you want to solve the equation $x/\pi + \sin(x) = 1$. You first enter the equation to make sure you have typed it properly.

  ```
  >  eq := x/Pi + sin(x) = 1;
  ```

  You can then plot the left hand and right hand sides of this equation, to see where they intersect. Notice the use of the commands lhs and rhs.

  ```
  >  plot([lhs(eq),rhs(eq)], x=-2*Pi..4*Pi);
  ```

  So there are three intersection points, i.e., solutions, one between 0 and 2, one between 2 and 4, and one between 4 and 6. To find the solutions, use Maple's fsolve command.

  ```
  >  fsolve(eq,x);
  ```

  This finds the solution between 2 and 4. (It's π.) The other two can be found by adding ranges to fsolve:

  ```
  >  fsolve(eq,x=0..2);    fsolve(eq,x=4..6);
  ```

  We now have all three solutions.

ASSIGNMENT: An ant is walking (to the right) over its ant mound, whose height (in cm) is given by the function: $h(x) = \dfrac{x^2 - 24x + 153928}{5(x^2 - 36x + 396)^2}$. Nearby there is a blade of grass, which is located as the line segment from $(40, 0.1)$ to $(40, 12)$. The goal in this lab is to find the point where the ant first sees the blade of grass. You can assume that the ant's line of sight is the tangent line to the ant mound. The following series of questions will lead you to the solution.

1. Define the function h that gives the height of the ant mound, using an arrow definition. Define the line segment occupied by the blade of grass and name it grass. Plot the ant mound in brown and the blade of grass in green on the same graph. Include the plot option, scaling=constrained.

2. Compute the derivative of h and name it Dh.

3. Compute the tangent line to $y = h(x)$ at $x = 16$ and define it as a function htan using an arrow definition. Plot the ant mound, the blade of grass, and the ant's line of sight when the ant is at $x = 16$. Can it see the blade of grass? Find the height H where the tangent line crosses the line $x = 40$ by evaluating htan(40.).

4. Compute the tangent line to $y = h(x)$ at $x = 17.5$ and define it as a function htan using an arrow definition. Plot the ant mound, the blade of grass, and the ant's line of sight when the ant is at $x = 17.5$. Can it see the blade of grass? Find the height H where the tangent line crosses the line $x = 40$ by evaluating htan(40.).

5. We can now see that, when the ant is at some position $x = a$ between 16 and 17.5, it can first see the top of the blade of grass. We want to find a. So, compute the tangent line to $y = h(x)$ at $x = a$ for a variable a. Define the tangent line at $x = a$ as a function htan using an arrow definition. NOTE: If you previously gave a a value, clear it by executing a:='a';

6. You can no longer plot the tangent line because its formula contains a variable, namely a. However, you can still find the height H where the tangent line crosses the line $x = 40$ by evaluating htan(40.). When this height H equals the height of the blade of grass, the ant can just begin to see the blade of grass. Use Maple's fsolve command to solve for the value of a where H equals the height of the blade of grass. (You may need to specify a range for a in the fsolve command.) Denote the solution by A.

7. For the value A found in problem 6, compute the tangent line to $y = h(x)$ at $x = A$ and define it as a function htan using an arrow definition. Plot the ant mound, the blade of grass, and the ant's line of sight when the ant is at $x = A$. Can it see the blade of grass? Find the height H where the tangent line crosses the line $x = 40$ by evaluating htan(40.).

8. There is a second solution to the equation $H = 12$. What is wrong with this solution?

9. *10% extra credit.* Animate the tangent line as the ant walks over the mound from $a = 7$ to $a = 19$. HINT: Use the animate command in the plots package.

12.6   Distance Between Two Curves

BACKGROUND: You need to know how to find the slope of the line between two points and the slope of a tangent line. You also need to know the condition on slopes which says that two lines are perpendicular to each other.

ASSIGNMENT: Given two curves that do not intersect, you want to find the shortest distance between the two curves. It is geometrically obvious (and can be justified mathematically) that the closest the curves come to each other are at the points P on the first curve and Q on the second curve such that the line from P to Q is perpendicular to both curves. More precisely, the line from P to Q is perpendicular to each of the tangnet lines at P and Q.

1. Start by considering the two curves $y = x^2$ and $y = \ln(x)$. Plot both curves on the same axes. Be sure to set `scaling=constrained` so that distances will be correct.

2. Let L_1 be the tangent line to $y = x^2$ at the point $P = (p, p^2)$.
   Let L_2 be the tangent line to $y = \ln(x)$ at the point $Q = (q, \ln(q))$.
   Let L_3 be the line between the points $P = (p, p^2)$ and $Q = (q, \ln(q))$.
   Find the slope m_1 of L_1, the slope m_2 of L_2, and the slope m_3 of L_3.

3. Write the equations which say L_3 is perpendicular to both L_1 and L_2.
   Solve these equations for p and q. Find P and Q.

4. Find the distance between the points P and Q.

5. Find the equations of the lines L_1, L_2 and L_3.
   Add the three lines to your plot.

6. Repeat steps 1-5 for each of the following pairs of curves:

   (a) $y = 2 + x^2$ and $y = 2x$

   (b) $y = e^x$ and $y = 2\ln(x)$

   (c) $y = \dfrac{1}{1 + x^2}$ and $y = x^2 + \dfrac{1}{x^2}$ in the first quadrant

7. (Optional) Write a Maple procedure `distbetwcrvs` whose arguments are two functions and plotting ranges for x and y, and which prints out the closest points P and Q, plots the functions, tangent lines and the line between P and Q and returns the shortest distance between the two functions. If you write a procedure, you may use it to do Part 6.

12.7   Tangent and Normal Lines

BACKGROUND: Know how to compute the tangent and normal lines to a curve. Recall that he tangent line at $x = a$ has slope $f'(a)$ and the normal line has slope $-1/f'(a)$.

ASSIGNMENT: Consider the tangent and normal lines to the curve $y = x^2$.
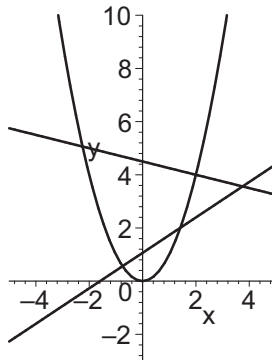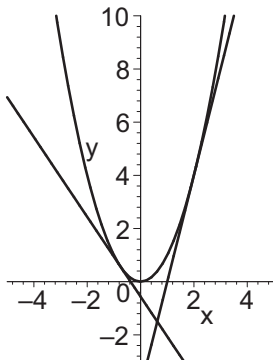
1. Find all points (p, q) which belong to exactly 2 distinct tangent lines to the curve $y = x^2$.

   HINT: Write the equation for the tangent line to $y = x^2$ at $x = a$. Write the equation which says (p, q) is on this line. This should be a quadratic equation to solve for a. When are there exactly 2 real solutions?

2. Find all points (p, q) which belong to exactly 2 (but not 3) distinct normal lines to $y = x^2$. Plot these points. This question is considerably harder than the first.

   HINT: Write the equation for the normal line to $y = x^2$ at $x = a$. Write the equation which says (p, q) is on this line. This can be rewritten as a cubic equation to solve for a. When are there exactly 2 real solutions? (A cubic equation $g(a) = 0$ has exactly 2 real roots only when there is a double root. This occurs when the double root is also a root of the derivative. So $g(a) = 0$ and $g'(a) = 0$ must both have a root at the same point.)

12.8   Parameterizing Letters

BACKGROUND: See Section 3.2 for a discussion of parametric curves. The goal
of this project is to use parametric curves to form letters. Letters can be made
with a combination of straight line segments, circles and ellipses which may be
parameterized as follows:

- The line segment from the point (x_1, y_1) to the point (x_2, y_2) is parameterized by $x = x_1 + t(x_2 - x_1)$ and $y = y_1 + t(y_2 - y_1)$ for $0 \leq t \leq 1$. For
  example, the following Maple commands plot the line segment from $(1, 0)$
  and $(2, 3)$:

  ```
  >  x1,y1:=1,0; x2,y2:=2,3;
  >  xt:=x1+t*(x2-x1); yt:=y1+t*(y2-y1);
  >  plot([xt,yt,t=0..1], 0..3, 0..3, scaling=constrained);
  ```

- An arc of a circle centered at (a, b) of radius r can be parameterized by
  $x = a + r\cos(t)$ and $y = b + r\sin(t)$ for t in an appropriate range of angles.
  For example, the following Maple commands plot a three-quarter circle of
  radius 2 centered at $(-1, 3)$.

  ```
  >  xt:=-1+2*cos(t); yt:= 3+2*sin(t);
  >  plot([xt,yt,t=Pi/4..7*Pi/4], -4..2, -1..5,
  >  scaling=constrained);
  ```

- Similarly, an arc of an ellipse, centered at the point (a, b) with horizontal
  radius r and vertical radius s is parameterized by $x = a + r\cos(t)$ and
  $y = b + s\sin(t)$.

- For example, the following commands plot the letter P:

  ```
  >  P1:=plot([0,2*t, t=0..1]):
  >  P2:=plot([0+.5*cos(t), 1.5+.5*sin(t), t=-Pi/2..Pi/2]):
  >  with(plots):
  >  display(P1,P2, scaling=constrained, axes=none);
  ```

ASSIGNMENT:

1. Plot the capital letters R and S, using straight line segments and arcs of
   circles or ellipses.

2. Repeat Exercise 1, but with the letters moved .5 units above the x axis
   and .5 units to the right of the y axis, and with its size doubled.

3. Use parametric curves to make graphs of the letters in your name.

12.9 Parametric Curves

GOAL: In this lab[1] you will use Maple to plot parametric curves, to find intersections of parametric curves with various lines, to find slopes of parametric curves and to find self-intersections of parametric curves.

BACKGROUND: Parametric curves and their Maple plots were discussed in Section 3.2. Recall from Section 2.3 that **fsolve** is best used in conjunction with a plot. If you're trying to find where a function equals zero, you should first plot it to make sure that **fsolve** is giving you all the roots, and to find the ranges to put into **fsolve**.

ASSIGNMENT: Consider the parametric curve

$$x = f(t) = 2\sin(2\pi t) - 2\cos^5(2\pi t) \qquad y = g(t) = \cos(2\pi t) - 3\sin(2\pi t)$$

for $0 \le t \le 1$. As you answer each question, be sure to refer back to the original plot to check that your answers make sense.

NOTE: "Find the points" with some property, means find the x and y coordinates.

1. Plot the curve with `scaling=constrained`. Find the points on the curve where t is $0, 0.1, 0.2, 0.3, 0.4, 0.5, 0.6, 0.7, 0.8$ and 0.9. Plot these 10 points and combine them with the plot of the curve. See Section 3.1 for point plots and combining plots. By hand (on a printed plot) or using `textplot`, label the points with t values.

2. Find the points where the curve crosses the line $x = 1$.
   HINT: You need to find the values of t which make the x coordinate equal to 1 by using `fsolve` and then use these t values to find the y coordinates. A plot of $f(t)$ as function a of t, may be helpful.

3. Find the points where the curve crosses the line $y = 2x$.
   HINT: Find the t values which solve $g(t) = 2f(t)$.

4. Find the points where the tangent line to the curve is horizontal.

5. Find the points where the tangent line to the curve is vertical.

6. Find the points where the tangent line has slope 3.
   HINT: This is the equation $\dfrac{dy}{dt} = 3\dfrac{dx}{dt}$.

7. Find the points where the curve crosses itself.
   HINT: You need to find two different values, $t = r$ and $t = s$, so that $f(r) = f(s)$ and $g(r) = g(s)$. Do this by using `fsolve` on this pair of equations, with ranges for r and s found from your plot of the curve. See the `?fsolve` examples.

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[1]This project was originally developed by Thomas Vogel, Texas A&M Univ.

12.10   Seeing a Blimp

ASSIGNMENT: A blimp[2] has the shape and location given by the ellipse

$$\left(\frac{x - 1000}{96}\right)^2 + \left(\frac{y - 1200}{30}\right)^2 = 1$$

which may be parametrized by

$$x = 1000 + 96\cos(\theta), \qquad y = 1200 + 30\sin(\theta).$$

What is the angular size in degrees of the blimp as seen from your eye which is located at the origin, $O = (0,0)$. In other words, find two points, $P = (a,b)$ and $Q = (c,d)$, on the blimp, so that the angle between the vectors $\overrightarrow{OP}$ and $\overrightarrow{OQ}$ is as large as possible. Plot the ellipse and the line segments $\overline{OP}$ and $\overline{OQ}$.
HINTS:

- The `angle` command in the `linalg` package can be used to find the angle in radians between two vectors. Only convert the final angle to degrees.

- Since the ellipse is longer in the x-direction, you would expect the points P and Q to be close to the endpoints of the x-diameter; i.e. close to the points $P_o = (904, 1200)$ and $Q_o = (1096, 1200)$. Find the angle between the vectors $\overrightarrow{OP_o}$ and $\overrightarrow{OQ_o}$. The actual angle should be slightly larger than this.

- If you do nothing to restrict the `fsolve` command, Maple is most likely to miss the maximum and find a minimum of 0 angle which occurs whenever $P = Q$. So in the `fsolve` command, restrict the ranges of the variables so that P is close to P_o and Q is close to Q_o.

- To solve the problem, first hold $Q = Q_o$ and vary $P = (a,b)$. Then hold $P = P_o$ and vary $Q = (c,d)$.

- If you solve the problem by eliminating a variables, express a in terms of b in the half of the ellipse near P_o and express c in terms of d in the half near Q_o.

- If you solve the problem by parametrizing the ellipse, then P is close to P_o when θ is close to π and Q is close to Q_o when θ is close to 0.

---

[2]The dimensions and altitude of the Goodyear blimp were taken from
http://www.goodyearblimp.com/q_construct.html#sizes   and
http://www.goodyearblimp.com/q_flying.html#high

12.11   Static and Dynamic Tension

BACKGROUND: Newton's Second Law says $\sum \vec{F} = m\vec{a}$, i.e. the sum of the vector forces on an object is equal to its mass times its acceleration vector. The situation is "static" when the object is not moving and $\vec{a} = \vec{0}$. It is "dynamic" when the object is moving and $\vec{a} \neq \vec{0}$.

ASSIGNMENT: A weight of mass $m = 10$ kg is supported at point $C = (x, y)$ by two stiff rods supported at points $A = (0,0)$ and $B = (z,0)$. Point A is held fixed while point B is free to slide horizontally along a rail. The rod AC has length $L_a = 3$ m. The rod BC has length $L_b = 5$ m. The force of gravity is $\vec{F}_g = (0, -mg)$ where $g = 9.8$ m/sec^2. The tension forces along the rods AC and BC are $\vec{T}_a = (-T_a \cos \alpha, T_a \sin \alpha)$ and $\vec{T}_b = (T_b \cos \beta, T_b \sin \beta)$ respectively.
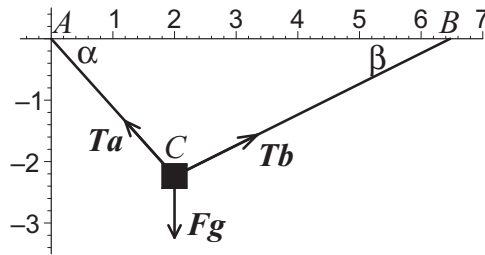
1. For what value of z is $\alpha = \pi/2$? What are the smallest and largest values of z? (NOTE: The angle α can be larger than $\pi/2$.) No Maple required.

2. The position of $B = (z, 0)$ determines α, β and the position of $C = (x, y)$. Set up and solve two equations for α and β as functions of z. Find parametric equations for x and y as functions of the parameter z. Plot x and y. Discuss the plots.

3. If the point $B = (z, 0)$ is held fixed, the situation is static. Set the sum of the forces equal to zero and solve for T_a and T_b, as functions of z. Plot T_a and T_b. What happens to T_a and T_b when $z = 4$? When $z < 4$? Give physical explanations.

4. When $B = (z, 0)$ moves, $C = (x, y)$ also moves. So the situation is dynamic. Assume $B = (z, 0)$ starts at $(2, 0)$ at $t = 0$ and moves to the right with constant velocity with $\dfrac{dz}{dt} = 1$ m/sec. Find $x(t)$, $\dfrac{dx}{dt}$, $\dfrac{d^2x}{dt^2}$, $y(t)$, $\dfrac{dy}{dt}$ and $\dfrac{d^2y}{dt^2}$. Set the sum of the forces equal to the mass times the acceleration and solve for T_a and T_b as functions of t. Then convert to functions of z. Plot T_a and T_b as functions of z. Discuss the acceleration and tension near $t = 2$, $(z = 4)$.

5. Compare the static and dynamic plots for T_a and T_b. Which are larger?